Received: *March 18, 1999*, Accepted: *April 12, 1999*

Non-static spherically symmetric solution of Einstein vacuum field equations with Λ

Amir H. Abbassi

Department of Physics, School of Sciences, Tarbiat Modarres University
P.O.Box 14155-4838, Tehran, Iran
E-mail: ahabbasi@net1cs.modares.ac.ir

Abstract: The Schwarschild-de Sitter metric is the known solution of Einstein field equations with cosmological constant term for vacuum spherically symmetric space around a point mass M. Recently it has been reported that in a Λ-dominant world the Schwarzschild type coordinate systems are disqualified by redshift-magnitude test as a proper frame of reference (gr-qc/9812092). We derive the solution in a FRW type coordinate system which is qualified according to the mentioned test. Asymptotically it approachs to the non-static form of deSitter metric. The obtained metric is transformable to Schwarzschild-de Sitter metric. It is an analytic function of r for all values except $r = 0$ which is singular. This is carried out with no making use of Eddington-Finkelstein coordinates and without entering any cross term in the metric.

Keywords: Classical Theories of Gravity, Black Holes.

JHEP04(1999)011

Today the best standard candles known in cosmology probably are type Ia supernovae. They have became the principle indicator for the determination of distance in the Hubble flow and consequently measuring more accurately the Hubble constant and the cosmological deceleration. The startling result of the recent high-z supernovae observation is that the deceleration parameter comes out negative, $q_0 \approx -1/2$. This implies an acceleration expansion rate. The following relation between q_0, Ω_M and Ω_Λ holds

$$q_0 = \frac{\Omega_M}{2} - \Omega_\Lambda \, , \tag{1}$$

which the values of $\Omega_M \approx 0.3$ and $\Omega_\Lambda \approx 0.7$ are favored by new achievements. A positive cosmological constant is inferred from recent measurements and indicate that cosmological constant is the dominant term in the Friedmann equation at our epoch [1, 2, 3, 4, 5]. So far in standard models of gravitating systems the existence of cosmological constant have been simply ignored,but after these observations the theoretical models should be modified to include this term as a part of reality.

A theoretical issue as has been well explained in standard reviews of cosmological constant is that the theoretical expectation for Λ exceeds its observational value by 120 orders of magnitude [6, 7]. Meanwhile it worth to mention that there is some way to resolve this deficiency. It is proved that there exists a different commutation relation between space coordinate and its momentum for massless particles [8]. The field theoretical counterpart of this new commutation relation leads to zero vacuum energy [9].

The familiar solution of the Einstein field equations with cosmological constant for a vacuum space around a center of spherical symmetry (SS) is given by the Schwarzschild-deSitter metric [10]. It has the following form ($G = c = 1$):

$$\begin{aligned} ds^2 = & \left(1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2\right) dt^2 - \left(1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2\right)^{-1} dr^2 - \\ & - r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right), \end{aligned} \tag{2}$$

where Λ stands for cosmological constant and M is the mass of an individual point mass which is the source of SS. This metric has two coordinate type singularity at distance very close to $2M - (M^3\Lambda)/2$ and $\sqrt{3/\Lambda} - M$ and an intrinsic singularity at $r = 0$. On the geodesic with $d\theta = d\varphi = 0$, ds^2 does change sign in passing from $2M - (M^3\Lambda)/3 < r < \sqrt{3/\Lambda} - M$ interval to the neighboring intervals $r < 2M - (M^3\Lambda)/3$ and $r > \sqrt{3/\Lambda} - M$. This enforces us to give up our common sense and accept the ad hoc character exchange of r and t as space and time coordinate. On the other hand according to the principle of causality we expect when M the source of SS tends to zero the symmetry of the metric which is the effect of the existence of M disappear, but in Schwarzschild-de Sitter metric the contrast of this happens and SS remains. The crucial deficiency of the Schwarzschild-de Sitter metric is that in a Λ-dominated universe the Schwarzschild type comoving coordinate system gives a redshift-magnitude relation which does not confirm with observational data [11]. This discards the Schwarzschild type coordinate as a proper frame of reference in the presence of Λ. A FRW type comoving coordinate system gives a correct prediction for redshift-magnitude relation which quite well agrees with observation.

We are going to find the solution of Einstein field equations with cosmological constant for a vacuum spherically symmetric space in a proper FRW type coordinate system. Although a system consisting a point mass M possesses SS, it is plausible to expect that at very large scales compared to M the space looks homogenous and isotropic within good approximaton and its related metric to be de Sitter metric in its FRW form. Thus it will not restrict the generality of the problem if take the following form for ds^2,

$$ds^2 = B(r,t)dt^2 - R^2(t)\left[A(r,t)dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right], \tag{3}$$

where A and B tends to one in the large distance limit and $R(t)$ the scale factor satisfies;

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{\ddot{R}}{R} = \frac{\Lambda}{3}. \tag{4}$$

Taking $ds^2 = -g_{\mu\nu}\, dx^\mu dx^\nu$, the nonvanishing components of $g_{\mu\nu}$ for metric (3) are:

$$\begin{aligned} g_{rr} &= R^2 A\,, & g_{\theta\theta} &= R^2 r^2 \\ g_{\varphi\varphi} &= R^2 r^2 \sin^2\theta\,, & g_{tt} &= -B\,. \end{aligned} \tag{5}$$

Since the off diagonal components of the metric are zero, its contravariant components are simply the inverse of the related covariant one.

Then the nonvanishing components of affine connection are:

$$\begin{aligned}
&\Gamma^r_{rr} = \frac{A'}{2A}, & &\Gamma^r_{rt} = \frac{\dot{R}}{R} + \frac{\dot{A}}{2A}, & &\Gamma^r_{\theta\theta} = -\frac{r}{A}, \\
& & &\Gamma^r_{\varphi\varphi} = -\frac{r\sin^2\theta}{A}, & &\Gamma^r_{tt} = \frac{B'}{2R^2 A} \\
&\Gamma^\theta_{r\theta} = \frac{1}{r}, & &\Gamma^\theta_{t\theta} = \frac{\dot{R}}{R}, & &\Gamma^\theta_{\varphi\varphi} = -\frac{\sin 2\theta}{2} \\
&\Gamma^\varphi_{r\varphi} = \frac{1}{r}, & &\Gamma^\varphi_{\theta\varphi} = \cot\theta\,, & &\Gamma^\varphi_{t\varphi} = \frac{\dot{R}}{R} \\
&\Gamma^t_{rr} = \frac{2\dot{R}RA + R^2\dot{A}}{2B}, & &\Gamma^t_{tr} = \frac{B'}{2B}, & &\Gamma^t_{\theta\theta} = \frac{r^2\dot{R}R}{B}, \\
& & &\Gamma^t_{\varphi\varphi} = \frac{r^2\sin^2\theta\,\dot{R}R}{B}, & &\Gamma^t_{tt} = \frac{\dot{B}}{2B}
\end{aligned} \tag{6}$$

where($'$) and ($\cdot$) represent the derivative with respect to r and t respectively.

The nonvanishing components of Ricci tensor in terms of the above affine connections are:

$$\begin{aligned}
R_{rr} &= \frac{B''}{2B} - \frac{\acute{A}}{rA} - \frac{\acute{B}}{4B}\left(\frac{\acute{B}}{B} + \frac{\acute{A}}{A}\right) - \\
&\quad - \frac{R^2 A}{B}\left[\frac{\ddot{R}}{R} + \frac{2\dot{R}^2}{R^2} + \frac{2\dot{R}\dot{A}}{RA} + \frac{\ddot{A}}{2A} - \frac{\dot{R}\dot{B}}{2RB} - \frac{\dot{A}\dot{B}}{4AB} - \left(\frac{\dot{A}}{2A}\right)^2\right] \qquad (7)\\
R_{tt} &= \frac{B}{R^2 A}\left[-\frac{B''}{2B} + \frac{\acute{B}}{4B}\left(\frac{\acute{B}}{B} + \frac{\acute{A}}{A}\right) - \frac{\acute{B}}{rB}\right] + \frac{3\ddot{R}}{R} + \frac{\ddot{A}}{2A} - \frac{\dot{A}^2}{4A^2} + \frac{\dot{R}\dot{A}}{RA} - \\
&\quad - \frac{3\dot{R}\dot{B}}{2RB} - \frac{\dot{A}\dot{B}}{4AB} \qquad (8)\\
R_{\theta\theta} &= -1 + \frac{1}{A} - \frac{r\acute{A}}{2A^2} + \frac{r\acute{B}}{2AB} + \frac{R^2 r^2}{B}\left[-\frac{2\dot{R}^2}{R^2} - \frac{\ddot{R}}{R} + \frac{\dot{R}\dot{B}}{2RB} - \frac{\dot{R}\dot{A}}{2RA}\right] \qquad (9)\\
R_{\varphi\varphi} &= \sin^2\theta\, R_{\theta\theta} \qquad (10)\\
R_{rt} &= -\frac{\dot{A}}{Ar} - \frac{\acute{B}\dot{R}}{BR}\,. \qquad (11)
\end{aligned}$$

The Einstein field equations with Λ for empty space are reduced to

$$R_{\mu\nu} = -\Lambda\, g_{\mu\nu}\,. \qquad (12)$$

With the use of (3) and (12) we may write

$$\frac{R_{rr}}{R^2 A} + \frac{R_{tt}}{B} = 0\,. \qquad (13)$$

Now inserting(7) and (8) in (13) results

$$-\frac{1}{R^2 rA}\left(\frac{\acute{A}}{A} + \frac{\acute{B}}{B}\right) - \frac{1}{B}\left(-\frac{2\ddot{R}}{R} + \frac{2\dot{R}^2}{R^2} + \frac{\dot{R}\dot{A}}{RA} + \frac{\dot{R}\dot{B}}{RB}\right) = 0\,. \qquad (14)$$

Using (4), (14) leads to

$$\frac{1}{R^2 rA}\left(\frac{\acute{A}}{A} + \frac{\acute{B}}{B}\right) + \frac{\dot{R}}{RB}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) = 0\,. \qquad (15)$$

Let us define ρ to be $\rho = R(t)\, r$ and assume

$$A(r,t) = A(\rho)\,, \qquad B(r,t) = B(\rho)\,. \qquad (16)$$

If (*) stands for the derivative with respect to ρ then the partial derivative of A and B with respect to r and t are

$$\begin{aligned}
\acute{A} &= R(t)A^*\,, \qquad \dot{A} = \dot{R}(t)rA^*\,,\\
\acute{B} &= R(t)B^*\,, \qquad \dot{B} = \dot{R}(t)rB^*\,. \qquad (17)
\end{aligned}$$

We may apply (17) to simplify (15) so it becomes

$$\left(\frac{1}{rAR}+\frac{r\dot{R}^2}{RB}\right)\left(\frac{A^*}{A}+\frac{B^*}{B}\right)=0\,. \tag{18}$$

Equation (18) holds if we have

$$\frac{A^*}{A}+\frac{B^*}{B}=0\,. \tag{19}$$

Integration of (19) with respect to ρ gives AB =constant, since A and B need to become one at large distances, thus we have

$$A=B^{-1}\,. \tag{20}$$

This selection automatically satisfies the rt-component of the field equation and may be considered as a justification for the assumption in (16). The $\theta\theta$-component of the field equation together with (4) and (20) yield

$$\left(-1+\frac{1}{A}\right)-\frac{r\acute{A}}{A^2}-\frac{\Lambda R^2 r^2}{B}-\frac{R\dot{R}r^2\dot{A}}{BA}=-\Lambda R^2 r^2 \tag{21}$$

and inserting (17) and (20) in (21) gives

$$\left(-1+\frac{1}{A}\right)-\frac{\rho A^*}{A^2}-\Lambda\rho^2(A-1)-\frac{\Lambda}{3}\rho^3 A^*=0$$

or

$$-1+\frac{d}{d\rho}\left(\frac{\rho}{A}\right)-\frac{\Lambda}{3}\frac{d}{d\rho}[\rho^3(\Lambda-1)]=0\,. \tag{22}$$

Integration of (22) with respect to ρ results

$$\frac{\rho}{A}-\frac{\Lambda}{3}\rho^3(\Lambda-1)-\rho=\text{constant}\equiv C\,. \tag{23}$$

If we define $D\equiv\rho A$, then (22) in terms of D becomes a quadratic equation

$$-\frac{\Lambda}{3}\rho D^2-\left(1-\frac{\Lambda}{3}\rho^2+\frac{C}{\rho}\right)D+\rho=0\,. \tag{24}$$

This quadratic equation has two real solutions:

$$D_\pm=\frac{\left(1-\frac{\Lambda}{3}\rho^2+\frac{C}{\rho}\right)\pm\sqrt{\left(1-\frac{\Lambda}{3}\rho^2+\frac{C}{\rho}\right)^2+\frac{4\Lambda\rho^2}{3}}}{-2\frac{\Lambda}{3}\rho}\,. \tag{25}$$

By evaluating the post Newtonian limits we observe that merely D_- is physically acceptable and the actual value of C is $-2M$. Thus we have

$$B=A^{-1}=\frac{2\Lambda\rho^2}{3}\left[\sqrt{\left(1-\frac{2M}{\rho}-\frac{\Lambda}{3}\rho^2\right)^2+\frac{4\Lambda}{3}\rho^2}-\left(1-\frac{2M}{\rho}-\frac{\Lambda}{3}\rho^2\right)\right]^{-1}\,. \tag{26}$$

Eq. (26) may be written in a more elegant form

$$B = A^{-1} = \frac{1}{2}\left[\sqrt{\left(1 - \frac{2M}{\rho} - \frac{\Lambda}{3}\rho^2\right)^2 + \frac{4\Lambda}{3}\rho^2} + \left(1 - \frac{2M}{\rho} - \frac{\Lambda}{3}\rho^2\right)\right]. \tag{27}$$

Thus the final non-static form of ds^2 in a FRW reference frame is

$$\begin{aligned} ds^2 = & \frac{1}{2}\left[\sqrt{\left(1 - \frac{2M}{\rho} - \frac{\Lambda}{3}\rho^2\right)^2 + \frac{4\Lambda}{3}\rho^2} + \left(1 - \frac{2M}{\rho} - \frac{\Lambda}{3}\rho^2\right)\right] dt^2 - \\ & -2e^{2\sqrt{\frac{\Lambda}{3}}t}\left[\sqrt{\left(1 - \frac{2M}{\rho} - \frac{\Lambda}{3}\rho^2\right)^2 + \frac{4\Lambda}{3}\rho^2} + \left(1 - \frac{2M}{\rho} - \frac{\Lambda}{3}\rho^2\right)\right]^{-1} dr^2 - \\ & -\rho^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) \\ \rho \equiv & \, e^{\left(\sqrt{\frac{\Lambda}{3}}t\right)} r\,. \end{aligned} \tag{28}$$

We may summarize the functional behavior of the obtained metric as follows. For the interval $\rho \gg 2M$ we may neglect the $2M/\rho$ term comparing to others and consequently come to $B = A^{-1} = 1$. This means that at these very large scales the essential symmetry of space restores. This property also guarantees that when M tends to zero the homogeneity and isotropy of space restore. For scales not too large but comparable to $2M$, (27) reduces to $B = A^{-1} = (1 - \frac{2M}{\rho} - \frac{\Lambda}{3}\rho^2)$ which exactly coincide with Schwarzschild-de Sitter metric. For the interval ρ quite smaller than $2M$ we have

$$\begin{aligned} B = A^{-1} & \approx \left\{\left|1 - \frac{2M}{\rho} - \frac{\Lambda}{3}\rho^2\right|\left[1 + \frac{\frac{2\Lambda}{3}\rho^2}{\left(1 - \frac{2M}{\rho} - \frac{\Lambda}{3}\rho^2\right)^2}\right] + \left(1 - \frac{2M}{\rho} - \frac{\Lambda}{3}\rho^2\right)\right\}\frac{1}{2} \\ & \approx \frac{\Lambda\rho^3}{6M}\,. \end{aligned} \tag{29}$$

The obtained metric does not change sign and has quite different behavior in the intervals which Schwarzschild-de Sitter metric faces with problems. In contrast to Schwarzschild-de Sitter metric that at origin gives $g_{tt} = +\infty$ and $g_{rr} = 0$, this new metric gives $g_{tt} = 0$ and $g_{rr} = +\infty$. thus the non-static metric is analytic everywhere except at $\rho = 0$. At $\rho = 2M - M^3\Lambda/3$ that is the familiar horizon of black holes the metric gives

$$A\left(2M - \frac{M^3\Lambda}{3}\right) \approx \sqrt{\frac{\Lambda}{3}} \times 2M\,. \tag{30}$$

On geodesic with $d\theta = d\varphi = 0$, the relation between dt and dr at this point for a massless particle is

$$dt = \frac{dr}{\sqrt{\frac{4}{3}\Lambda M^2}}\,. \tag{31}$$

Since Λ is of the order $10^{-56}cm^{-2}$, then it takes less than $\tau \equiv 10^5(\frac{M}{km})^{-1}$ years for a photon to escape from horizon region of black hole i.e. $\rho \approx 2M$ to outside of few M.

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Since the metric is nonsingular for $r \neq 0$ and there is no horizon, the high potential regions are accessible. This huge amount of potential energy can be converted to thermal or other physical forms of energy and providing us with a new supermachinary for high energy astrophysics. This can make drastic change in our understanding of the AGN phenomenon.

We should mention that the presented non-static metric and the Schwarzschild-de Sitter metric are transformable to each other by the following coordinate transformation

$$\rho = R(t)r \qquad T = t + \frac{\dot{R}}{R}\int^{\rho} \frac{A(\tilde{\rho})\tilde{\rho}d\tilde{\rho}}{1 - \frac{2M}{\tilde{\rho}} - \frac{\Lambda}{3}\tilde{\rho}^2}\,, \tag{32}$$

where A is given by (27). For the special case of $M = 0$, (32) reduces to

$$T = t - \frac{1}{2}\ln\left(1 - \frac{\Lambda}{3}\rho^2\right). \tag{33}$$

As it is evident from (33) this transformation does not preserve the symmetries of space and this is the reason that why in Schwarzschild-de Sitter metric the spherical symmetry remains in the limit $M \to 0$.

In a recent work I have shown that there exist general spherically symmetric solutions of Einstein vacuum field equations without Λ [12]. These solutions are free of intrinsic singularity and are analytic at $r = 0$. We anticipate that the combination of these two works will provide us the ultimate nonsingular metrics.

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